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Quantum Hall states on the cylinder as unitary matrix Chern-Simons theory

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Abstract: We propose a unitary matrix Chern-Simons model representing fractional quantum Hall fluids of finite extent on the cylinder. A mapping between the states of the two systems is established. Standard properties of Laughlin theory, such as the quantization of the inverse filling fraction and of the quasiparticle number, are reproduced by the quantum mechanics of the matrix model. We also point out that this system is holographically described in terms of the one-dimensional Sutherland integrable particle system.

Keywords: Chern-Simons Theories, Non-Commutative Geometry, Matrix Models, Integrable Hierarchies.

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1. Introduction

Chern-Simons actions for gauge fields, since their introduction to physics [1], have found numerous applications in elementary particle and condensed matter situations. In the last of these developments, Chern-Simons theory on the noncommutative plane has been proposed by Susskind as an effective description of the fractional quantum Hall fluid [2]. Specifically, the ground state of this theory can be interpreted as the Laughlin state for an infinite number of electrons [3]. The filling fraction corresponds to the inverse coefficient of the Chern-Simons action.

The basic idea behind this identification is that the noncommutative Chern-Simons action describes a (noncommutative) magnetic membrane, which, in turn, is equivalent to a magnetic fluid. The connection between (commutative) membranes and fluids is old [4, 5] and has recently been extended to supersymmetric fluids with spin [6]. The new ingredient in the quantum Hall connection is the proposal by Susskind that an essentially noncommutative fluid is appropriate in order to incorporate the discrete nature of electrons. This noncommutativity exists already at the classical level, and is distinct from the (quantum) noncommutativity of the coordinates of particles in the lowest Landau level [7, 8] and its extension to magnetohydrodynamics [9].

The above Chern-Simons theory can describe only an infinite number of electrons on an infinite plane. In a previous paper we proposed a regularized version of the noncommutative theory on the plane in the form of a Chern-Simons matrix model with boundary terms [10]. This model describes a system of finitely many electrons (a quantum Hall 'droplet') and was shown to reproduce all the relevant physics of the finite Laughlin states, such as boundary excitations, quantization of the filling fraction and quantization of the charge of quasiparticles (fractional holes). We further pointed out that the matrix model, and thus also the quantum Hall system, is equivalent to the Calogero model [11]–[15], a one-dimensional system of particles whose connection to fractional statistics [16, 17], anyons [18]–[21] and the quantum Hall system [22, 23] has long been established. An explicit (although non-unitary) mapping between the states of the matrix model and Laughlin states was presented in [24].

It is of interest to extend the correspondence between the noncommutative Chern-Simons matrix model and the quantum Hall system for spaces of different topologies which have compact dimensions. This is of both theoretical and practical significance, since compact spaces provide a natural regularization and have been used in alternative approaches to Laughlin states.

In this paper we shall present such a generalization for the case of a space with cylindrical topology, in the form of a unitary matrix model. We shall identify and analyze its classical and quantum states and establish a mapping to Laughlin states. Similarly to the planar case, we shall demonstrate that this model is equivalent to a periodic version of the Calogero model known as the Sutherland model. Finally we shall conclude with some directions for future research.

2. Chern-Simons theory on the noncommutative plane and quantum Hall states

Before analyzing the problem on the cylinder, we will review the basic features of Chern-Simons (CS) theory on a noncommutative plane and a commutative time and its connection with quantum Hall states, as proposed by Susskind [2]. We will also briefly review the corresponding Chern-Simons matrix model describing finite quantum Hall droplets [10].

2.1 Noncommutative Chern-Simons theory from magnetic fluids

The system to be described consists of an incompressible fluid of $N \to \infty$ spinless electrons on the plane in an external constant magnetic field B (we take their charge $e = 1$). Their coordinates are represented by two (infinite) hermitian matrices X_a, $a = 1, 2$, that is, by two operators on an infinite Hilbert space. The average electron density is $\rho_0 = 1/2\pi\theta$. The action is the analog of the gauge action of particles in a magnetic field:

$$S = \int dt \, \frac{B}{2} \, \mathrm{Tr} \left\{ \epsilon_{ab} (\dot{X}_a + i[A_0, X_a]) X_b + 2\theta A_0 \right\} \tag{2.1}$$

with Tr representing (matrix) trace over the Hilbert space and $[.,.]$ representing matrix commutators. The above has the form of a noncommutative CS action in the operator formulation [25]. Gauge transformations are conjugations of X_a by arbitrary time-dependent unitary operators which are compact enough to leave traces invariant. These have nontrivial topology and lead to level quantization [26] (see [27] for an analysis of this class of transformations). In the quantum Hall context they take the meaning of reshuffling the labels of the electrons, a generalization of particle permutation operators. Equivalently, the X_a can be considered as coordinates of a two-dimensional fuzzy membrane, $2\pi\theta$ playing the role of an area quantum and gauge transformations realizing area preserving diffeomorphisms.

The time component of the gauge field ensures gauge invariance, its equation of motion imposing the Gauss law constraint

$$-iB \, [X_1, X_2] = B\theta = \frac{1}{\nu} \tag{2.2}$$

with $\nu = 2\pi\rho_0/B$ being the filling fraction. The canonical conjugate of X_1 is $P_2 = BX_2$, so the operator in the left-hand side of (2.2) is the generator of gauge transformations on X_a. Since gauge transformations are interpreted as reshufflings of particles, (2.2) above has the interpretation of endowing the particles with quantum statistics of order $1/\nu$.

We will assume that X_1, X_2 provide an irreducible representation of the Gauss law (2.2), else we would be describing multiple layers of quantum Hall fluids. This representation is essentially unique, modulo gauge transformations, so there is a unique state in this theory (the vacuum). Deviations from the vacuum state can be achieved by introducing sources in the action [2]. A localized source at the origin has a density of the form $\rho = \rho_0 - q\delta^2(x)$ in the continuous (commutative) case, representing a point source of particle number $-q$, that is, a hole of charge q for $q > 0$. The noncommutative analog of such a density is

$$[X_1, X_2] = i\theta(1 + q|0\rangle\langle 0|) \, , \tag{2.3}$$

where $|n\rangle$, $n = 0, 1, \ldots$ is an oscillator basis for the (matrix) Hilbert space, $|0\rangle$ representing a state of minimal spread at the origin. In the membrane picture the right-hand side of (2.3) corresponds to area and implies that the area quantum at the origin has been increased to $2\pi\theta(1+q)$, therefore piercing a hole of area $A = 2\pi\theta q$ and creating a particle deficit $q = \rho_0 A$. We shall call this a quasihole state. For $q > 0$ a solution of (2.3) is

$$X_1 + iX_2 = \sqrt{2\theta} \sum_{n=1}^{\infty} \sqrt{n+q}\, |n-1\rangle\langle n| \,. \tag{2.4}$$

The above assumes that $|0\rangle$ is really a state at the origin, meaning $(X_1+iX_2)|0\rangle = 0$. Without this residual condition (2.3) has many more solutions, since $|0\rangle$ is a gauge-dependent state that can be reshuffled around. For instance, a class of solutions is

$$\tilde{X}_1 + i\tilde{X}_2 = P\sqrt{2\theta} \sum_{n=1}^{\infty} \sqrt{n + q\vartheta(n-n_o)}\, |n-1\rangle\langle n|\, P \tag{2.5}$$

with $\vartheta(s)$ the usual step function ($\vartheta(s) = 1$ if $s > 0$, else $\vartheta(s) = 0$) and P the permutator of $|0\rangle$ and $|n_o\rangle$

$$P = 1 - (|0\rangle - |n_o\rangle)(\langle 0| - \langle n_o|) \,. \tag{2.6}$$

This represents an annular hole of charge q at distance $\sim \sqrt{\theta n_o}$ from the origin.

For quasiparticles ($q < 0$), as long as $-q < 1$ we have a similar equation and solution. For $-q > 1$, however, clearly equations (2.3) and (2.4) cannot hold since the area quantum cannot be diminished below zero. The correct equation is, instead,

$$[X_1, X_2] = i\theta \left(1 - \sum_{n=0}^{k-1} |n\rangle\langle n| - \epsilon |k\rangle\langle k| \right) , \tag{2.7}$$

where k and ϵ are the integer and fractional part of the quasiparticle charge $-q$. The solution of (2.7) is

$$X_1 + iX_2 = \sum_{n=0}^{k-1} z_n |n\rangle\langle n| + \sqrt{2\theta} \sum_{n=k+1}^{\infty} \sqrt{n-k-\epsilon}\, |n-1\rangle\langle n| \,. \tag{2.8}$$

where again we assumed that $(X_1 + iX_2)|k\rangle = 0$, so $|k\rangle$ now represents the state at the origin. In the membrane picture, k quanta of the membrane have 'peeled' and occupy positions $z_n = x_n + iy_n$ on the plane, while the rest of the membrane has a deficit of area at the origin equal to $2\pi\theta\epsilon$, leading to a charge surplus ϵ. The quanta are electrons that sit on top of the continuous charge distribution. If we want all charge density to be concentrated at the origin, we must choose all $z_n = 0$, which means that $X_1 + iX_2$ annihilates all $|n\rangle$ for $n = 0, 1 \ldots k$.

More general quasihole (particle) states, with the holes (or fractional part of the particles) positioned at arbitrary points on the plane can easily be constructed, but we shall not do so here. We point out that the above particle states for integer q are identical to flux solitons of noncommutative gauge theory [28]–[31].

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2.2 Finite Chern-Simons matrix model

For a finite number of electrons N we take the coordinate X_a to be finite hermitian $N \times N$ matrices. The action (2.1), however, and the Gauss law (2.2) are inconsistent for finite matrices, and a modified action must be written. We take [10]

$$S = \int dt \frac{B}{2} \mathrm{Tr} \left\{ \epsilon_{ab} (\dot{X}_a + i[A_0, X_a]) X_b + 2\theta A_0 - \omega X_a^2 \right\} + \Psi^\dagger (i\dot{\Psi} - A_0 \Psi) \qquad (2.9)$$

Ψ is a complex N-vector that transforms in the fundamental of the gauge group $U(N)$:

$$X_a \to U X_a U^{-1} , \qquad \Psi \to U \Psi \qquad (2.10)$$

while the term proportional to ω serves as a spatial regulator providing a harmonic potential that keeps the electrons near the origin.

The Gauss law now reads

$$G \equiv -iB\, [X_1, X_2] + \Psi\Psi^\dagger - B\theta = 0 \, . \qquad (2.11)$$

Taking the trace of the above equation gives

$$\Psi^\dagger \Psi = NB\theta \, . \qquad (2.12)$$

The equation of motion for Ψ in the $A_0 = 0$ gauge is $\dot{\Psi} = 0$. So we can take it to be

$$\Psi = \sqrt{NB}\, |v\rangle \, , \qquad (2.13)$$

where $|v\rangle$ is a constant vector of unit length. So the traceless part of (2.11) reads

$$[X_1, X_2] = i\theta \, (1 - N|v\rangle\langle v|) \, . \qquad (2.14)$$

This is similar to (2.2) for the infinite plane case, with an extra projection operator, which is the minimal deformation of the planar result (2.2) that has a vanishing trace. Ψ clearly acts like a boundary term, absorbing the 'anomaly' of the commutator $[X_1, X_2]$, very much like the case of a boundary field theory required to absorb the anomaly of a bulk CS field theory.

The classical states of this theory are given by the set of matrices $A = X_1 + iX_2$ satisfying (2.14), and can be explicitly found [10]. The ground state is found by minimizing the potential $V = (B\omega/2)\mathrm{Tr}(X_1^2 + X_2^2)$ while imposing the constraint (2.14). We obtain the solution

$$X_1 + iX_2 = \sqrt{2\theta} \sum_{n=0}^{N-1} \sqrt{n} |n-1\rangle\langle n| \, , \qquad |v\rangle = |N-1\rangle \, . \qquad (2.15)$$

This is essentially a quantum harmonic oscillator projected to the lowest N energy eigenstates. The radius squared matrix $R^2 = X_1^2 + X_2^2$ has a finite, equidistant spectrum. So the above solution represents a circular quantum Hall 'droplet' of radius $\sqrt{2N\theta}$ and particle density $\rho_0 = N/(\pi R^2) \sim 1/(2\pi\theta)$ as in the infinite plane case.

Excitations of the classical ground state can be considered. A class of such excitations are perturbations of $A = X_1 + iX_2$ generated by the infinitesimal transformation

$$A' = A + \sum_{n=0}^{N-1} \epsilon_n (A^\dagger)^n \tag{2.16}$$

with ϵ_n infinitesimal complex parameters. The sum is truncated to $N-1$ since $A^\dagger$ is an $N \times N$ matrix and only its first N powers are independent. These map the boundary of the droplet to the new boundary (in polar coordinates)

$$R'(\phi) = \sqrt{2N\theta} + \sum_{n=-N}^{N} c_n e^{in\phi}\,, \tag{2.17}$$

where the coefficients c_n are

$$c_n = c_{-n}^* = \frac{R^n}{2}\epsilon_{n-1} \qquad (n>0)\,, \qquad c_0 = 0\,. \tag{2.18}$$

This is an arbitrary area-preserving deformation of the boundary of the droplet, truncated to the lowest N Fourier modes. The above states are, therefore, arbitrary area-preserving boundary excitations of the droplet [32, 33, 34], appropriately truncated to reflect the finite, noncommutative nature of the system.

A second class of excitations are the analogs of quasihole and quasiparticle states. States with a quasihole of charge $-q$ at the origin are of the form

$$A = \sqrt{2\theta}\left(\sqrt{q}|N\rangle\langle 0| + \sum_{n=1}^{N-1}\sqrt{n+q}|n-1\rangle\langle n|\right)\,, \qquad q>0 \tag{2.19}$$

representing a circular droplet with a circular hole of area $2\pi\theta q$ at the origin, that is, with a charge deficit q. Note that (2.19) stills respects the Gauss constraint (2.14) (with $|v\rangle = |N-1\rangle$) without the explicit introduction of any external source. The hole and the boundary of the droplet together cancel the anomaly of the commutator, the outer boundary part absorbing an amount $N+q$ and the inner (hole) boundary producing an amount q.

Fractional quasiparticle states cannot be written in this model, reflecting the fact that such states do not belong to the $\nu = 1/B\theta$ Laughlin state. Particle states with an integer particle number $-q = m$ and the extra m electrons positioned outside the droplet do exist, but we shall not write their explicit form here.

We conclude by pointing out that boundary excitations, quasiholes and particle state can all be continuously deformed to each other, due to the finite number of degrees of freedom of the model. Such transformations become highly nonperturbative in the $N \to \infty$ limit.

3. A model for finite number of electrons on the cylinder

The proposed model works well for electrons on an infinite plane. For a space representing a cylinder of radius R we take one of the coordinates, say, X_2, to be periodic with period $2\pi R$. Clearly the above model does not take into account this periodicity and has to be suitably modified in order to correctly describe the physics in the compact dimension. We shall propose here such a model, appropriate to describing one compact dimension.

There are two routes for achieving this. The first is to write a matrix model on the covering space and then reduce it [35, 36], leading to matrices depending on an additional continuous parameter dual to the compact dimension, that is, a field theory. The second is to represent the compact dimensions with unitary matrices [37]. As we shall demonstrate, the two approaches turn out to be equivalent in our case.

3.1 The Chern-Simons unitary matrix model

We shall begin with the second approach, which is simpler and leads more directly to the desired model. The main point is that the coordinate x_2 is not single-valued on the toroidal space and thus is not a physical observable. An alternative coordinate which is single-valued on the cylinder is the exponential $e^{ix_2/R}$. For a noncommutative space we define the unitary operator

$$U = e^{iX_2/R}. \tag{3.1}$$

Together with the hermitian operator $X_1 \equiv X$, they parametrize a noncommutative cylinder. The planar noncommutativity relation for X_1, X_2 translates into

$$UXU^{-1} = X + \frac{\theta}{R}. \tag{3.2}$$

To write the Chern-Simons action on such a space we imitate again the magnetic action of a particle with coordinates X_i. In the Landau gauge $A_1 = 0$, $A_2 = -BX_1$ the lagrangian reads $-BX_1\dot{X}_2$. Representing $\dot{X}_2$ as $-iRU^{-1}\dot{U}$ and including a Lagrange multiplier A_0 which reproduces (3.2), the full action becomes

$$S = \int dt\, B\, \mathrm{Tr}\left\{ iR\, U^{-1}(\dot{U} + i[A_0, U])X + \theta A_0 \right\}. \tag{3.3}$$

It is again expressed in terms of a covariant time derivative $D_0 U$ of the unitary operator U. Note that, in the above, we have adopted the ordering $U^{-1}\dot{U}$ for the operator representing $\dot{X}_2$. Had we adopted the ordering $\dot{U}U^{-1}$ we would have ended with an action involving $D_0 U\, U^{-1} X$ instead of $U^{-1} D_0 U X$. The two actions represent identical physics, upon redefining $X \to UXU^{-1}$ (which, upon use of the noncommutativity relation (3.2), is simply a shift $X \to X + (\theta/R)$).

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To describe a finite system consisting of N electrons we need to take the coordinates to be finite $N \times N$ matrices. The constraint (3.2), however, is not consistent for finite matrices, just as in the planar case, and we need to modify the action with 'boundary' terms that render a consistent form for the constraint. For our purposes we take

$$S = \int dt\, B\, \mathrm{Tr}\left\{ iRU^{-1}(\dot{U} + i[A_0, U])X + \theta A_0 - \frac{\omega}{2}X^2 \right\} + \Psi^\dagger(i\dot{\Psi} - A_0\Psi)\,. \quad (3.4)$$

It is similar to the planar Chern-Simons matrix model in [10]. The boundary term involves Ψ, a complex N-vector boson that transforms in the fundamental of the gauge group U(N). Its role is to absorb the 'anomaly' of the group commutator $UXU^{-1} - X$, analogous to a boundary field theory required to absorb the anomaly of a bulk CS field theory. We also added a spatial regulator term in the form of a harmonic oscillator potential in the direction of the cylinder's axis. Since the U-direction is compact we need not worry about localizing particles there, and we only want to localize them in the infinite X-direction.

Before analyzing this model further, we present the alternative derivation in terms of the covering space reduction and demonstrate that it produces a model equivalent to the model proposed above.

3.2 The Chern-Simons matrix field theory

An alternative approach to deriving the desired matrix model is to augment the dimensionality of the hermitian matrices X_1, X_2 of the planar model in [10] from N to pN and take $p \to \infty$. p represents the copies of the cylinder on the (planar) covering space. To ensure that the state is the same on all copies we must impose the condition that fields in different copies are gauge equivalent; that is, the operator which shifts copies is a unitary (gauge) transformation. Therefore, there should exist some unitary matrix U, representing shifts by one copy, satisfying

$$\begin{aligned} UX_1U &= X_1\,, & UX_2U^{-1} &= X_2 + 2\pi R\,, \\ UA_0U^{-1} &= A_0\,, & U\Psi &= e^{i\alpha}\Psi\,. \end{aligned} \quad (3.5)$$

This can be explicitly realized by parametrizing the indices I, J of $(X_a)_{IJ}$, $(A_0)_{IJ}$ and Ψ_I as

$$I = i + nN\,, \qquad i = 1, \ldots N\,, \qquad n = \ldots -1, 0, 1, \ldots \quad (3.6)$$

That is, split X_a in terms of $N \times N$ blocks, with the diagonal (n, n) blocks representing the electrons on the n-th copy in the covering space and the off-diagonal (n, m) blocks representing effective 'interactions' of electrons between the n-th and m-th copies on the covering space. Clearly the (n, n) copy must be the same as the $(0, 0)$ copy, only shifted by $2\pi Rn$ in the X_2 direction. Further, the interactions between the m and n copies must only depend on the distance between the copies $m - n$. Similarly, the

Lagrange multiplier $(A_0)_{m,n}$ must impose constraints on the m and n copies that depend only on their distance $m-n$. Finally, Ψ_n, representing the boundary of the state in copy n, must be the same for all n, up to an irrelevant phase. Overall we have

$$\begin{aligned} (X_1)_{m,n} &= (X_1)_{m-n}\,, & (X_2)_{m,n} &= (X_2)_{m-n} + 2\pi R n \delta_{mn} \\ (A_0)_{m,n} &= (A_0)_{m-n}\,, & \Psi_n &= e^{in\alpha}\Psi\,. \end{aligned} \tag{3.7}$$

The unitary transformation U is simply the shift $n \to n+1$.

We see that the matrices X_a and A_0 are now parametrized in terms of an additional integer n. Hermiticity of X_a and A_0 in the original indices I, J means

$$(X_a)_n = (X_a^\dagger)_{-n}\,, \qquad (A_0)_n = (A_0^\dagger)_{-n}\,. \tag{3.8}$$

We can, therefore, define the Fourier transforms

$$X_a(\sigma) = \sum_n (X_a)_n e^{in\sigma}\,, \qquad A_0(\sigma) = \sum_n (A_0)_n e^{in\sigma} \tag{3.9}$$

with σ a variable with periodicity 2π. $X_a(\sigma)$ and $A_0(\sigma)$ are hermitian σ-dependent $N \times N$ matrices. Matrix multiplication in the original I, J indices translates into matrix multiplication pointwise in σ, while I-trace translates into σ-integration and matrix trace. It is also useful to define

$$\Psi(\sigma) = \sum_n \Psi_n e^{in\sigma} = \Psi \delta(\sigma + \alpha)\,. \tag{3.10}$$

We can now write the original matrix model (with a confining harmonic potential in the X_1 direction)

$$S = \int dt\, B\, \mathrm{Tr}\left\{ -X_1(\dot{X}_2 + i[A_0, X_2]) + \theta A_0 - \frac{\omega}{2} X_1^2 \right\} + \Psi^\dagger(i\dot{\Psi} - A_0\Psi) \tag{3.11}$$

in terms of the matrices $X_i(\sigma), A_0(\sigma)$. A standard calculation leads to the result

$$\begin{aligned} S = &\int dt d\sigma\, B\, \mathrm{Tr}\left\{ -X_1(\dot{X}_2 + i[A_0, X_2] - R\partial_\sigma A_0) - \frac{\omega}{2} X_1^2 + \theta A_0 \right\} + \\ &+ \int dt\, \Psi^\dagger(i\dot{\Psi} - A_0(\sigma = \alpha)\Psi)\,. \end{aligned} \tag{3.12}$$

This is nothing but $1+1$-dimensional U(N) Yang-Mills theory with a Wilson line source at $\sigma = \alpha$ and a uniform background charge. To see this, rename $t = \sigma_0$, $R^{-1}\sigma = \sigma_1$, $X_1 = -F/\omega$, $X_2 = A_1$. Then the above action becomes

$$S = \int d^2\sigma \frac{RB}{\omega} \mathrm{Tr}\left(F F_{01} - \frac{1}{2} F^2 + B\omega\theta A_0 \right) + \int dt\, \Psi^\dagger(i\dot{\Psi} - A_0(\sigma = \alpha)\Psi)\,, \tag{3.13}$$

where we defined the field strength

$$F_{01} = \partial_0 A_1 - \partial_1 A_0 + i[A_0, A_1]\,. \tag{3.14}$$

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We recognize the Yang-Mills action in the first-order formalism, on a circular space of radius R^{-1}. In addition, there is a constant background U(1) charge density $B\omega\theta$ and a localized source at $\sigma_1 = \alpha$ depending on Ψ. The latter corresponds to an insertion of a Wilson line in the temporal direction carrying the direct sum of all symmetric representations of the gauge group U(N).

To make this last point explicit, consider the temporal direction σ_0 compact and euclidean with period T. This also turns A_0 into iA_0. An appropriate gauge transformation can render $A_0(\sigma_0, \alpha)$ diagonal and independent of σ_0. The diagonal elements $(A_0)_{jj}$ (no sum in j) at $\sigma_1 = \alpha$ correspond to the eigenvalues of the temporal Wilson line element at $\sigma_1 = \alpha$, $e^{i\lambda_j}$, specifically

$$\left(Pe^{i\int_0^T A_0 dt}\right)_{jj} = e^{i\lambda_j} = e^{i(A_0)_{jj}T} \qquad (\text{no sum in } j)\,. \tag{3.15}$$

In this gauge the components of Ψ decouple and become N independent bosonic harmonic oscillators with frequencies $i(A_0)_{jj}$. Integrating out Ψ produces the partition function of these N oscillators which, assuming normal ordering, is

$$\begin{aligned}\prod_j \frac{1}{1-e^{i(A_0)_{jj}T}} &= \prod_j \left(1 + e^{i\lambda_j} + e^{2i\lambda_j} + \cdots\right) \\ &= 1 + \sum_j e^{i\lambda_j} + \left(\sum_j e^{i2\lambda_j} + \sum_{j<k} e^{i\lambda_j + i\lambda_k}\right) + \cdots\,. \end{aligned} \tag{3.16}$$

We recognize the terms in the last sum as the trace of the temporal Wilson loop in the singlet, fundamental, doubly symmetric etc. representations. The above is, then, a (bosonic) oscillator representation of a Wilson loop element, reproducing all symmetric representations. A discussion of arbitrary representations in the context of the planar CS matrix model can be found in [38].

It is a remarkable fact that the above theory can be reduced to a unitary matrix model identical to the one derived in the previous section. The details are explained in [39]. The basic point is that two-dimensional Yang-Mills theory has no propagating modes and the only dynamical degrees of freedom are in the nontrivial holonomy of the gauge field A_1 around the spatial direction. Defining the U(N) line element

$$U[a,b] = Pe^{i\int_a^b A_1 d\sigma_1} \tag{3.17}$$

the phase space variables in the gauge $A_0 = 0$ are

$$U = U[\alpha, 2\pi + \alpha]\,, \qquad X = -\frac{1}{\omega}\int_\alpha^{2\pi+\alpha} U[\alpha,\sigma]\; FU[0,\sigma]^{-1} d\sigma\,. \tag{3.18}$$

In terms of these the action reduces to

$$S = \int dt\, B\, \mathrm{Tr}\left\{iRU^{-1}\dot{U}X - \frac{\omega}{2}X^2\right\} + i\Psi^\dagger\dot{\Psi}\,. \tag{3.19}$$

The Gauss law for the fields A_1 and F involves the background charge θ and the point source Ψ at $\sigma_1 = \alpha$. Expressed in terms of the reduced phase space variables it becomes the constraint

$$X - UXU^{-1} = \frac{1}{RB}\Psi\Psi^\dagger - \frac{\theta}{R}\,. \tag{3.20}$$

Inserting the above constraint in the action through a new field A_0 (which is now only a function of time) we recover the unitary Chern-Simons matrix model (3.4). Thus, both routes to compactifying X_2 lead to the proposed unitary matrix model.

4. Classical states of the unitary matrix model

4.1 General solution

To get a feeling of the physics of the above model we shall analyze its classical structure and states. We can again impose the A_0 equation of motion as a Gauss constraint and then put $A_0 = 0$. In our case it reads

$$G \equiv RB\,(UXU^{-1} - X) + \Psi\Psi^\dagger - B\theta = 0\,. \tag{4.1}$$

The trace of the above equation gives as in the planar case

$$\Psi^\dagger\Psi = NB\theta\,. \tag{4.2}$$

In the $A_0 = 0$ gauge Ψ is constant, take it $\sqrt{NB}\,|v\rangle$ with $|v\rangle$ a constant unit vector. The traceless part of (4.1) reads

$$UXU^{-1} - X = \frac{\theta}{R}\,(1 - N|v\rangle\langle v|)\,. \tag{4.3}$$

This is similar to (3.2) for the infinite cylinder, with an extra projection operator. Again, using the residual time-independent $U(N)$ gauge freedom to rotate $|v\rangle$ to the form $|v\rangle = (0, \ldots 0, 1)$ we obtain the form $(\theta/R)\,diag\,(1, \ldots, 1, 1-N)$ for the above constraint, which is the minimal deformation of the result (3.2) that has a vanishing trace.

The equations of motion for X and U read

$$\dot{X} + [U^{-1}\dot{U}, X] = 0\,, \qquad iRU^{-1}\dot{U} - \omega X = 0 \tag{4.4}$$

or

$$\dot{X} = i\frac{R}{\omega}\frac{d}{dt}(U^{-1}\dot{U}) = 0\,. \tag{4.5}$$

This represents free motion on the manifold $U(N)$, as represented by the unitary matrix $U(t)$, with X playing the role of matrix momentum. It is solved by

$$U(t) = U_0 e^{-i\omega X_0 t/R}\,, \qquad X(t) = X_0\,, \tag{4.6}$$

where the constant matrices X_0, U_0 satisfy the constraint (4.3). We can find all the classical states of this model by diagonalizing $U_0 = \mathrm{diag}\,\{e^{i\phi_n}\}$. By examining the diagonal and off-diagonal elements of (4.3) we see that the components of $|v\rangle$ have length $|v_n|^2 = 1/N$. Choosing their phases as $v_n = \exp(-i\phi_n/2)/\sqrt{N}$, as we can do using the residual $U(1)^N$ gauge invariance, we obtain

$$(U_0)_{mn} = e^{i\phi_n}\delta_{mn}\,, \qquad (X_0)_{mn} = x_n\delta_{mn} + \frac{i\theta}{2R\sin(\frac{\phi_m-\phi_n}{2})}(1-\delta_{mn})\,. \tag{4.7}$$

The solution is parametrized by the N compact eigenvalues of U_0, ϕ_n, and the N diagonal elements of X, x_n, that is, by N coordinates on the cylinder.

4.2 Classical ground state

The lowest energy state, that is, the state most closely packed around $X_1 = 0$, is found by minimizing the potential $(B\omega/2)X^2$ while respecting the constraint (4.3). Implementing it with a matrix Lagrange multiplier Λ we obtain

$$B\omega X + U^{-1}\Lambda U - \Lambda = 0\,, \qquad [X, U^{-1}\Lambda U] = 0\,. \tag{4.8}$$

The above is solved by

$$\begin{aligned} U_{mn} &= e^{i\phi_0}\delta_{m,n+1}\,, \qquad X_{mn} = \frac{\theta}{R}\left(\frac{N+1}{2} - n\right)\delta_{mn} \\ \Lambda_{mn} &= \frac{\theta B\omega}{2R}\left(\frac{N+1}{2} - n\right)\left(\frac{N+1}{2} - n + 1\right)\delta_{mn}\,, \qquad v_n = \delta_{n,1}\,. \end{aligned} \tag{4.9}$$

The eigenvalues of U are $\exp(i\phi_0 + i2\pi n/N)$ and are evenly distributed on the compact dimension of the cylinder. Similarly, the eigenvalues of X are evenly distributed along the axis of the cylinder and span a length $\sim N\theta/R$. Therefore, the above solution represents a tubular quantum Hall droplet around the cylinder with an area $A = 2\pi R \cdot N\theta/R$ and an average density $\rho_0 = N/A = 1/(2\pi\theta)$.

We point out that the average distance between successive electrons in the x_1 direction is θ/R. In the planar case, particles were evenly distributed on the plane with a density $1/2\pi\theta$ and thus an average distance of order $\sqrt{\theta}$. This is a signal that, on the cylinder, quantum Hall states do not have a constant density. In the extreme case $\theta \gg R^2$ the electrons will behave more like one-dimensional particles with well-defined positions along the length of the cylinder. We also point out the existence of ϕ_0 in the solution for U, which does not affect the energy.

4.3 Equivalence to the Sutherland model

Just as in the planar case, the matrix model above is equivalent to a one-dimensional particle system, the so-called Sutherland model [12]. This is an integrable system of N nonrelativistic particles on the circle with coordinates and momenta (ϕ_n, p_n) and

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hamiltonian

$$H = \sum_{n=1}^{N} \frac{\omega}{2B} p_n^2 + \sum_{n \neq m} \frac{\nu^{-2}}{4 \sin^2 \frac{\phi_n - \phi_m}{2}} . \tag{4.10}$$

It can be thought of as the Calogero model of particles on the line with inverse-square mutual potentials, rendered periodic in space with period $2\pi R$. In terms of the parameters of the model, the mass of the particles is B/ω and the coupling constant of the two-body inverse-square potential is ν^{-2}. We refer the reader to [14, 15, 40] for details of the derivation of the connection between the matrix model and the Sutherland model and to [39, 41] for the connection between 2-dimensional Yang-Mills and the Sutherland model. Here we simply state the relevant results and give their connection to quantum Hall quantities.

The positions of the Sutherland particles on the circle ϕ_n are the eigenvalues of U, while the momenta p_n are the diagonal elements of X_1, specifically $p_n = Bx_n$. The motion of the ϕ_n generated by the hamiltonian (4.10) is compatible with the evolution of the eigenvalues of U as it evolves in time according to (4.5). So the Sutherland model gives a holographic description of the quantum Hall state by monitoring the effective electron coordinates along X_2, that is, the eigenvalues of U.

The hamiltonians of the Sutherland and matrix model are equal and energy states map. The ground state is obtained by putting the particles at their static equilibrium positions. Because of their repulsion and the symmetry of the problem, they will form a uniform lattice of points on the circle, as in (4.9). ϕ_0 is the coordinate of the center of mass of the particles. Sound waves on this lattice correspond to small perturbations of the quantum Hall state. As the amplitude grows large, they become nonlinear nondispersive waves corresponding to holes forming in the quantum Hall particle distribution along the X-direction. In the limit they become solitons, representing isolated particles off the quantum Hall ground state [42]. Further connections at the quantum level will be described in subsequent sections.

5. Quantization of the cylindrical matrix Chern-Simons model

5.1 Gauss law and quantization of the filling fraction

We now proceed with the quantization of the above model. We use double brackets for quantum commutators and double kets for quantum states to distinguish them from matrix commutators and N-vectors.

The canonical structure of (3.4) implies the Poisson brackets

$$\{X_{ij}, U_{kl}\} = \frac{i}{iBR} \delta_{il} U_{kj} . \tag{5.1}$$

This means that $-BRX$ is the generator of right-rotations of the matrix U. Indeed, for any hermitian matrix ϵ

$$\{-BR\,\mathrm{tr}(\epsilon X), U\} = iU\epsilon\,. \tag{5.2}$$

Quantum mechanically X should also generate right-rotations of U. In the U-representation, then, it acquires the form

$$X_{ij} = -\frac{1}{BR}\,U_{kj}\frac{\partial}{\partial U_{ki}}\,. \tag{5.3}$$

As a result, $\mathcal{R} \equiv -BRX$ satisfies the $U(N)$ algebra:

$$[[\mathcal{R}_{ij}, \mathcal{R}_{kl}]] = \delta_{il}\mathcal{R}_{kj} - \delta_{kj}\mathcal{R}_{il} \tag{5.4}$$

with the obvious hermiticity condition $X_{ij}^\dagger = X_{ji}$. Similarly, the classical matrix $\mathcal{L} \equiv BRUXU^{-1}$ generates left-rotations of U. It should therefore be ordered as

$$\mathcal{L}_{ij} = -U_{ik}\frac{\partial}{\partial U_{jk}} \tag{5.5}$$

and also satisfies the $U(N)$ algebra (5.4). The sum of these operators $G_U \equiv \mathcal{L} + \mathcal{R}$ satisfies the $SU(N)$ algebra ($\mathcal{L}$ and $\mathcal{R}$ have equal and opposite $U(1)$ parts) and generates unitary conjugations of the matrix U.

The components of Ψ are harmonic oscillators, satisfying

$$[[\Psi_i, \Psi_j^\dagger]] = \delta_{ij}\,. \tag{5.6}$$

The matrix $G_\Psi \equiv \Psi\Psi^\dagger$ generates rotations of the vector Ψ, and must also satisfy the $U(N)$ algebra (5.4). It should therefore be ordered as

$$(G_\Psi)_{ij} = \Psi_j^\dagger\Psi_i\,. \tag{5.7}$$

The Gauss law constraint then acquires the form

$$(\mathcal{L} + \mathcal{R} + G_\Psi - \theta B)|phys\rangle\rangle = 0\,. \tag{5.8}$$

The situation is similar to the planar case. G_U contains only symmetric products of the adjoint, with a number of boxes in their Young tableau (Z_N charge) a multiple of N, kN (k integer). G_Ψ contains only totally symmetric representations with $U(1)$ charge equal to the number of boxes in the Young tableau. For the total representation to be in the singlet, as required by the Gauss law, G_U and G_Ψ must be in conjugate representations and thus G_Ψ must also have a number of boxes kN. Moreover, the trace ($U(1)$ charge) of G_Ψ must cancel $N\theta B$. So we obtain the quantization condition

$$B\theta = k\,, \qquad k = \text{integer}\,. \tag{5.9}$$

Again, this is related to the level quantization of the noncommutative Chern-Simons action [26, 43] and can also be attributed to a global gauge anomaly of the model [40]. The above condition will lead to the quantization of the inverse filling fraction, as in Laughlin theory. We anticipate the actual result as $\nu = 1/(k+1) \equiv 1/n$; the shift from k to $k+1$ is a quantum correction.

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5.2 Quantum states

The hamiltonian is

$$H = \frac{B\omega}{2}\,\mathrm{tr}X^2 = \frac{\omega}{2BR^2}\,\mathrm{tr}\mathcal{L}^2 = \frac{\omega}{2BR^2}\,\mathrm{tr}\mathcal{R}^2\,. \tag{5.10}$$

It is the laplacian on the group manifold U(N) (remember that X is essentially the momentum of U), and is proportional to the common quadratic Casimir of $\mathcal{L}$ or $\mathcal{R}$.

Since the space U(N) is curved, we could add to the hamiltonian a term proportional to the curvature. Such terms can always arise through quantum ordering effects; a particular value, equal to 1/8 times the curvature, is singled out from conformal invariance. In our case the curvature is constant and, as we shall see, the addition of a constant term as above will make the spectrum especially simple and suggestive.

Quantum states can be represented in terms of wavefunctions of U. A particular set of such wavefunctions diagonalizes H. Specifically, consider the matrix elements $R_{\alpha\beta}(U)$ of the matrix U in some irreducible representation of SU(N) R of dimension d_R. By Schur's lemma, any arbitrary function of U can be expanded in terms of the above functions.

Each matrix element $R_{\alpha\beta}$ above is, in fact, an eigenstate of H. To see this, note that under arbitrary left- and right-rotations of U the states transform as

$$R_{\alpha\beta}(V^{-1}UW) = R^{-1}_{\alpha\gamma}(V)R_{\gamma\delta}(U)R_{\delta\beta}(W)\,. \tag{5.11}$$

So the multiplet $R_{\alpha\beta}(U)$, $\alpha,\beta = 1,\ldots d_R$ transforms in the R representation under right-rotations of U and in the conjugate $\bar{R}$ representations under left-rotations of U. Since H is the quadratic Casimir of $\mathcal{R}$ or $\mathcal{L}$ we obtain

$$HR_{\alpha\beta}(U) = \frac{\omega}{2BR^2}\,C_{2,R}\,R_{\alpha\beta}(U) = E_R\,R_{\alpha\beta}(U)\,. \tag{5.12}$$

So the spectrum of H consists of all quadratic Casimirs $C_{2,R}$, each with a degeneracy d_R^2 corresponding to the different matrix elements of $R_{\alpha\beta}(U)$.

We still need to impose the Gauss law constraint. According to the discussion of the previous section, it stipulates that the states for U transform in a totally symmetric representation S_k, with kN boxes, under conjugations of U. This means that we must pick the corresponding representation for $G_U = \mathcal{L}+\mathcal{R}$. Clearly the d_R^2 states $R_{\alpha\beta}(U)$ transform in the $R\times\bar{R}$ representation under G_U. We must, therefore, decompose $R\times\bar{R}$ into irreducible components and pick the symmetric representation S_k among the components. Each S_k corresponds to a unique physical state (the components within S_k are contracted in a unique way with the components of the Ψ-representation S_k to give a singlet). This fixes the degeneracy of the eigenvalue E_R to the number of times $D_{R,k}$ that S_k appears in $R\times\bar{R}$.

The above takes care of the SU(N) part of the wavefunction. We can always assign an arbitrary U(1) part by multiplying the wavefunction with $(\det U)^q$. If we want single-valuedness under the transformation $U \to \exp(i2\pi)\, U$, which corresponds to rotations around the cylinder, we must have $q = p/N$ with p an integer, corresponding to integer U(1)-charge of the state under $\mathcal{L}$ or $\mathcal{R}$. Since the U(1) charge of R is the number of boxes, we can simply take the N rows of its Young tableau to have either positive or negative length.

We have reduced the physical spectrum of the model to pure group theory. In the present case, however, we can do better than that. Using standard Young tableau multiplication rules, it can be verified that in order to obtain S_k in the product $R \times \bar{R}$, the representation R must have lengths of Young tableau rows ℓ_j that satisfy

$$\ell_j - \ell_{j+1} \geq k\,, \qquad j = 1, \ldots N\,. \tag{5.13}$$

For each such R, S_k is contained exactly once in $R \times \bar{R}$.

The next interesting fact is that the quadratic Casimir of U(N) can be expressed in terms of the spectrum of free fermions on the circle [44]. Specifically, define the fermion 'momenta'

$$p_j = \ell_j + \frac{N+1}{2} - j\,, \qquad j = 1, \ldots N \tag{5.14}$$

which satisfy $p_j > p_{j+1}$. Then the expression for $C_{2,R}$ is

$$C_{2,R} = \sum_{j=1}^{N} \left(p_j^2 - p_{j,0}^2\right), \tag{5.15}$$

where $p_{j,0}$ are the 'ground state' momenta, corresponding to the singlet representation $\ell_j = 0$:

$$p_{j,0} = \frac{N+1}{2} - j\,. \tag{5.16}$$

The final observation is that if we add to the hamiltonian a curvature term with coefficient 1/8, as mentioned in the beginning of the section, the 'ground state' term in $C_{2,R}$ exactly cancels. We are left, then, with the spectrum of free identical particles on the circle, but with an enhanced exclusion principle. Specifically

$$E_R = \sum_j E_j = \frac{\omega}{2BR^2} \sum_j p_j^2\,, \tag{5.17}$$

where E_j are effective single-particle pseudo-energies, and the p_j are single-particle pseudomomenta. Because of (5.14) and (5.13), the p_j obey

$$p_j - p_{j+1} \geq k + 1 = n\,. \tag{5.18}$$

We obtain the spectrum of free nonrelativistic particles of mass ω/B on a circle of radius R, obeying exclusion statistics of order $n = k + 1$. This is the spectrum of Sutherland particles, which we have recovered entirely in the matrix model context.

It is reasonable to interpret E_j as the quantum analogs of the eigenvalues of the potential $(B\omega/2)X^2$. This means that the positions of the electrons along the X-direction are

$$x_j = \frac{1}{BR} p_j \,. \tag{5.19}$$

The ground state quasimomenta

$$p_{j,gs} = n\frac{N+1-2j}{2}\,, \tag{5.20}$$

form a 'Fermi sea' with distance n between successive momenta. The x_j, then, correspond to evenly spaced electrons with a distance $d = n/BR$ between them, and therefore an average density $\rho = 1/(2\pi R\cdot d)$. The filling fraction then is

$$\nu = \frac{2\pi\rho}{B} = \frac{1}{n} \tag{5.21}$$

justifying the interpretation of $n = k+1$ as the quantized inverse filling fraction. $k = 0$ (the singlet sector) corresponds to free fermions, reproducing the fully filled $n = 1$ Landau level.

Quasiparticle and quasihole states are identified in a way completely analogous to the planar case. A quasiparticle state is obtained by peeling a 'particle' from the surface of the sea (quasimomentum $p_{1,gs}$) and putting it to a higher value $p_1 > n(N-1)/2$. This corresponds to an electron at position $x \sim p_1/B$ along X in a state covariant under rotations of the cylinder.

Quasiholes correspond to the minimal excitations of the ground state inside the quantum Hall tubular distribution. This is achieved by leaving all quasimomenta p_j for $j \geq r$ unchanged, for some integer r, and increasing all p_j, $j < r$ by one unit:

$$\begin{aligned} p_j &= n\frac{N+1-2j}{2}, \qquad j \geq r \\ &= n\frac{N+1-2j}{2} + 1 \qquad j < r\,. \end{aligned} \tag{5.22}$$

This increases the gap between p_r and p_{r+1} to $n+1$ and creates a minimal 'hole' at position $x \sim p_{r,gs}/BR$. As in the planar case, removal of one particle corresponds to the creation of n holes, and therefore the particle number of the hole is $-q = -1/n = -\nu$. We again recover the quantization of the quasihole charge in fundamental units of

$$q_h = \nu = \frac{1}{n} \tag{5.23}$$

in accordance with Laughlin theory.

Finally, we have center-of-mass (U(1)) excitations, achieved by shifting all p_j by the same amount. This corresponds to translations of the electron state along the axis of the cylinder.

6. Correspondence to Laughlin states

The discussion in the last section demonstrates that there is a qualitative mapping between Laughlin states and matrix states. It is desirable to establish a more precise and explicit mapping between the two systems.

The wavefunction of the electrons in the lowest Landau level is a function of two variables. Alternatively, we can use the reduced phase space representation, in which spatial coordinates do not commute and span a quantum phase space. The mapping between the two representation in the plane is through coherent states. Laughlin states can then be considered as particular restricted many-body Landau wavefunctions, or corresponding states in the reduced phase space.

In order to establish the mapping between matrix and Laughlin states we will first define coherent states on a cylindrical phase space, then establish the correspondence with Laughlin states on the cylinder and finally map to matrix model states.

6.1 Coherent states on a cylindrical phase space

On a planar quantum phase space with coordinates X, Y satisfying $[X, Y] = i/B$ we can define creation and annihilation operators $a, a^\dagger$ as $a = (X + iY)\sqrt{B/2}$. Coherent states, then, are defined as eigenstates of the annihilation operator a and represent states of minimum uncertainty.

On a cylindrical phase space with one compact coordinate, say, $Y \equiv Y + 2\pi R$, the operator Y is multivalued and thus unphysical. The above creation-annihilation operators are therefore unphysical and we cannot use them to define coherent states. The operator $\exp[(X + iY)/R]$, however, is single-valued and physical. We will then define coherent states $|z\rangle$ by the relation

$$e^{(X+iY)/R}|z\rangle = e^{z/R}|z\rangle \,. \tag{6.1}$$

We can easily find the expression for the wavefunction of $|z\rangle$. In the Y representation, wavefunctions are periodic functions of Y and can be expanded in terms of the usual momentum states $|n\rangle$:

$$\langle Y|n\rangle = e^{inY/R} \,, \qquad n = \ldots, -1, 0, 1, \ldots \tag{6.2}$$

In this representation X acts as $(i/B)\partial/\partial Y$. A straightforward calculation shows that $|z\rangle$ is of the form

$$|z\rangle = N \sum_n e^{-\frac{nz}{R} - \frac{n^2}{2BR^2}} \, |n\rangle \,, \tag{6.3}$$

where N is a normalization factor. Writing $z = x + iy$, we see that $|z\rangle$ is a state with X centered around x (although X has discrete eigenvalues) and Y centered around y (modulo $2\pi R$).

It is convenient to choose the normalization N as

$$N = (2\pi^{3/2}B^{1/2}R)^{-1/2}e^{-Bx^2/2} = N_o e^{-Bx^2/2} . \tag{6.4}$$

Then it is straightforward to verify the completeness relation

$$\int dz d\bar{z}\, |z\rangle\langle z| = 1 . \tag{6.5}$$

Finally, the coherent state wavefunction of an n-state is

$$\langle z|n\rangle = N_o\, e^{i\frac{ny}{R} - \frac{(n+BRx)^2}{2BR^2}} \tag{6.6}$$

or, defining $w = \exp(-x + iy)$ as the corresponding analytic coordinate on the cylinder,

$$\langle w|n\rangle = N_o\, w^n e^{-\frac{B}{2}x^2 - \frac{n^2}{2BR^2}} . \tag{6.7}$$

6.2 Laughlin states on the cylinder

We start by presenting the single-particle wavefunctions for the lowest Landau level on the cylinder. These are well-known, and can easily be found in the Landau gauge $A_x = 0$, $A_y = -Bx$. In a basis diagonalizing the momentum in the Y-direction they become

$$\langle x, y|n\rangle_L \equiv \psi_n(z = x + iy) = e^{i\frac{ny}{R} - \frac{B}{2}\left(x + \frac{n}{BR}\right)^2} = w^n e^{-\frac{B}{2}x^2 - \frac{n^2}{2BR^2}} . \tag{6.8}$$

They are 'stripe' states, exponential in the y-direction and gaussian in the x-direction with a center shifted by $-n/BR$.

We immediately see the similarity with the coherent states in the reduced phase space. We conclude that Landau wavefunctions in terms of coordinates on the cylinder equal the corresponding coherent states on the reduced phase space.

N-body Laughlin states on the cylinder can be defined in a way analogous to the plane. There, the electrons were restricted to states containing the 'ground state' factor

$$\psi_{gs} = \prod_{j<k}(z_j - z_k)^n e^{-\frac{B}{2}\sum_j |z_j|^2} . \tag{6.9}$$

For $n = 1$ this is the Slater determinant of the lowest N angular momentum eigenstates, which reduces to the Vandermonde determinant for the variables z_i times the (non-analytic) N-body oscillator ground state. For higher n the analytic (Vandermonde) part is raised to the n-th power, while the ground state part remains the same.

A similar construction can be repeated on the cylinder, but with the momentum around the cylinder replacing the angular momentum. The corresponding Laughlin wavefunction would be

$$\psi_{gs} = \prod_{j<k}(w_j - w_k)^n e^{-\frac{B}{2}\sum_j x_j^2} . \tag{6.10}$$

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The relevant single-particle states appearing in (6.11) are, now, (6.8). The lowest y-momentum appearing in (6.11) is 0, while the highes is $n(N-1)$, corresponding to positions in the x-direction from 0 to $-n(N-1)/BR$. So this correspond to a tubular Hall state with area $A \sim 2\pi n(N-1)/B$, corresponding to a filling fraction $\nu = 1/n$.

To minimize the potential in x, the state should be centered around $x = 0$ and thus the above state should be shifted to the right by $\Delta x = n(N-1)/2BR$. This amounts to multiplying the wavefunction by the shift factor $\prod_j w_j^{-n(N-1)/2}$. The resulting, properly centered Laughlin wavefunction is

$$\psi_{gs} = \prod_{j<k} (u_{jk})^n e^{-\frac{B}{2}\sum_j x_j^2} , \qquad u_{jk} = \left(\frac{w_j}{w_k}\right)^{1/2} - \left(\frac{w_k}{w_j}\right)^{1/2} . \tag{6.11}$$

For small distances the above wavefunction has the same behavior as the planar Laughlin state. It is interesting that its probability density also admits a plasma interpretation, as in the planar case. Indeed, the Coulomb potential on a torus takes the form

$$V(x,y) = \frac{1}{2}\ln\left(\sinh\frac{z}{2R}\sinh\frac{\bar{z}}{2R}\right) = \frac{1}{2}\ln\left(w^{1/2} - w^{-1/2}\right) + \frac{1}{2}\ln\left(\bar{w}^{1/2} - \bar{w}^{-1/2}\right) . \tag{6.12}$$

Therefore, the Vandermonde part of $|\psi_{gs}|^2$ in (6.11) will reproduce the exponential of the mutual Coulomb potential of particles on the torus, while the gaussian part represents the potential of a 'neutralizing' constant background charge distribution, in exact analogy to the planar case.

The above ground state wavefunction can just as well be interpreted as the coherent wavefunction on the reduced phase space. Excited states above the ground state can be written by multiplying this wavefunction by any totally symmetric wavefunction of the w_i. Such wavefunctions can be uniquely expressed in terms of the Schur basis

$$S_{c_1,c_2,\ldots} = \left(\sum_j w_j^{c_i}\right)\left(\sum_k w_k^{c_2}\right)\cdots \tag{6.13}$$

and the general state will be a product of $S_{c_1,c_2\ldots}$ and ψ_{gs}:

$$\psi_{c_1,c_2\ldots} = \left(\sum_j w_j^{c_i}\right)\left(\sum_k w_k^{c_2}\right)\cdots\prod_{j<k}(w_j - w_k)^n e^{-\frac{B}{2}\sum_j x_j^2} , \tag{6.14}$$

where we omitted the shift factor since, as all symmetric functions, it can be reproduced in terms of Schur functions.

6.3 Mapping to matrix states

The states of the matrix model can be explicitly written in a way analogous to the one for the plane [24]. We will work in the U representation for the matrices and the oscillator representation for the Ψ. Define a ground state wavefunction $|0\rangle$ which is the Fock vacuum of the oscillators Ψ and the singlet (constant) in U; that is,

$$\Psi_j|0\rangle = X_{jk}|0\rangle = 0\,. \tag{6.15}$$

Excited states can be obtained by applying $\Psi^\dagger$'s and U's on $|0\rangle$. For the resulting state to be gauge invariant, all indices of U_{jk} and $\Psi_j^\dagger$ must be contracted, either with each other or with the SU(N) antisymmetric tensor $\epsilon^{j_1 j_2 .. j_N}$.

The U(1) gauge constraint (5.9), on the other hand, stipulates that each physical state should have exactly kN operators $\Psi^\dagger$. The minimal way that we can contract the indices of the above $\Psi^\dagger$'s is

$$|gs\rangle = \left(\epsilon^{j_1 .. j_N} \Psi^\dagger_{j_1} (\Psi^\dagger U)_{j_2} \cdots (\Psi^\dagger U^{N-1})_{j_N}\right)^k |0\rangle\,. \tag{6.16}$$

This can be considered the ground state of the system. The powers of U that appear must clearly be all different, and we chose them to span the values $0, 1, \ldots N-1$. We could have, instead, chosen the values $-\frac{N-1}{2}, \ldots \frac{N-1}{2}$, which would have given the U part of the state a vanishing U(1) charge. This can always be done a posteriori by multiplying with $\det U^{-(N-1)/2}$, and we shall stay with $|gs\rangle$ above for simplicity.

Other states can be obtained by multiplying with gauge invariant combinations of U's (no more $\Psi^\dagger$ are allowed). These are spanned by the Schur functions

$$S_{c_1,c_2\ldots} = \mathrm{tr} U^{c_1} \mathrm{tr} U^{c_2} \cdots \tag{6.17}$$

and a complete basis for the matrix states is

$$|c_1, c_2 \ldots\rangle = \mathrm{tr} U^{c_1} \mathrm{tr} U^{c_2} \cdots \left(\epsilon^{j_1 .. j_N} \Psi^\dagger_{j_1} (\Psi^\dagger U)_{j_2} \cdots (\Psi^\dagger U^{N-1})_{j_N}\right)^k |0\rangle\,. \tag{6.18}$$

The next step is to parametrize U in terms of diagonal and angular variables:

$$U = V^{-1} \Lambda V \tag{6.19}$$

with V an SU(N) matrix and $\Lambda = \mathrm{diag}\{e^{i\phi_j}\} \equiv diag\{W_j\}$ the eigenvalues of U. Each ϵ-factor in the states (6.18) becomes

$$\begin{aligned} &\epsilon^{j_1 .. j_N} \Psi^\dagger_{j_1} (\Psi^\dagger U)_{j_2} \cdots (\Psi^\dagger U^{N-1})_{j_N} = \\ &\quad = \epsilon^{j_1 .. j_N} (\Psi^\dagger V^{-1})_{k_1} V_{k_1 j_1} (\Psi^\dagger V^{-1})_{k_2} W_{k_2} V_{k_2 j_2} \cdots (\Psi^\dagger V^{-1})_{k_N} W_{k_N}^{N-1} V_{k_N j_N} \\ &\quad = \left\{\epsilon^{j_1 .. j_N} V_{k_1 j_1} \cdots V_{k_N j_N}\right\} \left\{W_{k_1}^0 \cdots W_{k_N}^{N-1}\right\} \left\{(\Psi^\dagger V^{-1})_{k_1} \cdots (\Psi^\dagger V^{-1})_{k_N}\right\}\,. \end{aligned} \tag{6.20}$$

Since the ϵ tensor antisymmetrizes the indices j_n, the indices k_n appearing in $V_{k_n j_n}$ in the first bracket are also antisymmetrized and we obtain

$$\epsilon^{j_1 .. j_N} V_{k_1 j_1} V_{k_2 j_2} \cdots V_{k_N j_N} = \epsilon^{k_1 \ldots k_N} \det V = \epsilon^{k_1 \ldots k_N} \tag{6.21}$$

($\det V = 1$ since V is special unitary). Defining

$$\chi = V \Psi \tag{6.22}$$

the expression (6.20) becomes

$$\epsilon^{k_1\ldots k_N}\, W^0_{k_1}\cdots W^{N-1}_{k_N}\, \chi^\dagger_{k_1}\cdots\chi^\dagger_{k_N}\,. \tag{6.23}$$

Since all k_n are distinct, the product of the $\chi^\dagger_{k_n}$ is simply $\chi^\dagger_1\chi^\dagger_2\cdots\chi^\dagger_N$. The remaining W's with the ϵ symbol reproduce the Vandermonde determinant. So the expression above becomes

$$\prod_{j<m}(W_j - W_m)\prod_j \chi^\dagger_j\,. \tag{6.24}$$

The Schur functions, on the other hand, become

$$\mathrm{tr}U^c = \sum_j W^c_j\,. \tag{6.25}$$

Overall, the states of the matrix model (6.18) take the form

$$|c_1, c_2\ldots\rangle = \left(\sum_j W^{c_1}_j\right)\left(\sum_j W^{c_2}_j\right)\cdots\prod_{j<m}(W_j - W_m)^k\left(\prod_j\chi^\dagger_j\right)^k|0\rangle\,. \tag{6.26}$$

The operators χ_j defined above are also harmonic oscillators and they satisfy $[[\chi_j,\chi^\dagger_k]] = \delta_{jk}$. So the oscillator state

$$|\Omega\rangle = \left(\prod_j\chi^\dagger_j\right)^k|0\rangle \tag{6.27}$$

appearing above has a norm $\langle\Omega|\Omega\rangle$ independent of the matrix V which enters in the definition the χ_j. Since no other oscillator state can ever appear, the above state, used for calculations of matrix elements, is effectively independent of V and W_j. The matrix V has, therefore, completely disappeared from the picture.

When calculating matrix elements between states, we must integrate with the Haar measure $[dU]$ over the matrix U. This is

$$[dU] = [dV]\prod_{j<k}|W_j - W_k|^2\prod_j d\phi_j\,. \tag{6.28}$$

The integration $[dV]$ over V produces a constant, since nothing depends on V. We are left with an integration over the eigenvalues ϕ_j, with an additional Vandermonde term coming from the measure. It is convenient to incorporate this measure into the definition of the states, so that matrix elements can be calculated with a flat measure over the ϕ_j. This introduces an additional power of the Vandermonde determinant in (6.26), shifting k to $k+1=n$. This is the origin of the renormalization of the filling fraction that we mentioned before. The final matrix states are, therefore,

$$|c_1, c_2\ldots\rangle = \left(\sum_j W^{c_1}_j\right)\left(\sum_j W^{c_2}_j\right)\cdots\prod_{j<m}(W_j - W_m)^n|\Omega\rangle\,. \tag{6.29}$$

These are identical in form to the corresponding Laughlin states (6.14) upon mapping $W_j = e^{i\phi_j}$ to w_j and identifying the ground state $|0\rangle$ with the bosonic gaussian factor in (6.14).

The above provides a formal mapping between the states of the unitary Chern-Simons matrix model and Laughlin states on the cylinder, much like the one for the plane [24]. It should be stressed, however, that the above mapping is *not* unitary. Indeed, the wavefunctions (6.29) above are integrated with a flat metric in ϕ_j, while the Laughlin states (6.14) are integrated with the planar measure $dxdy$. Therefore, the norms and scalar products of the two sets of states are, in general, different. This mapping is, therefore, at best a qualitative one. The exact, unitary mapping between matrix model states and Laughlin states, planar or cylindrical, is still lacking. Some relevant results in the planar case are derived in [45].

Note, further, that the above states (6.29) are not eigenstates of the eigenstates of the matrix model hamiltonian $\mathrm{tr}X^2$. Eigenstates can always be constructed by forming appropriate linear combinations of states (6.29) with the same degree of homogeneity, it essentially amounts to constructing the characters of the appropriate allowed representations of $\mathrm{U}(N)$ (see the discussion in section 5.2).

7. Outlook

We have extended a previous proposal and presented a unitary matrix Chern-Simons model describing fractional quantum Hall states of N electrons on the cylinder. The correspondence of the two systems was established as a map between states in each Hilbert space. As in the planar case, the quantization of the inverse filling fraction and of the quasihole charge are straightforward consequences of the quantum mechanics of this model. We also stressed that the classical value of the inverse filling fraction is shifted quantum mechanically by one unit. This can be equivalently viewed as a renormalization of the Chern-Simons coefficient, as a group-theoretic effect or as a result of the nontrivial measure of the model.

We further pointed out that this model, and therefore also the two-dimensional quantum Hall system, is described holographically in terms of a one-dimensional system, the so-called Sutherland integrable model of particles on the circle.

The correspondence between states and operators of the matrix and quantum Hall systems is still an open issue. In principle, such a correspondence is guaranteed to exist, since the two Hilbert spaces have the same dimensionality, as demonstrated in the last section. For it to be useful, however, it should be such that operators in the quantum Hall system map into explicit, simple matrix model operators. The hope is that the operators in the matrix model language will not involve explicitly the filling fraction (unlike, e.g., the hole creation operators in the second-quantized fractional quantum Hall system), and thus will describe the properties of these systems in a more universal way. This would also open the road for the calculation of relevant quantities, such as correlation functions, in the matrix model formulation. This is a most important issue.

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There are clearly many other open questions. Incorporating the spin of the electrons and identifying skyrmion-like configurations is an obvious next step. Most intriguing, however, is the question of a possible phase transition of the quantum Hall system at small filling fractions. Numerical simulations suggest that at $\nu^{-1} \sim 67$ Laughlin electrons form a Wigner crystal instead of an incompressible fluid. This is based on the properties of the Laughlin wavefunctions and does not seem to hinge on the specific dynamics of the electrons beyond what is already encoded in the wavefunctions themselves. The corresponding one-dimensional Sutherland system does not exhibit any such phase transition. This does not guarantee, however, that two-dimensional quantities calculated in the context of this model would not exhibit nonanalytic (or at least crossover) behavior in the filling fraction, signaling a phase transition. This intriguing possibility is the subject of further research.

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References

[1] S. Deser, R. Jackiw and S. Templeton, *Three-dimensional massive gauge theories*, *Phys. Rev. Lett.* **48** (1982) 975; *Topologically massive gauge theories*, *Ann. Phys. (NY)* **140** (1982) 372.

[2] L. Susskind, *The quantum hall fluid and non-commutative Chern Simons theory*, hep-th/0101029.

[3] R.B. Laughlin, *The quantum hall effect*, R.E. Prange and S.M. Girvin eds.

[4] J. Goldstone, unpublished.

[5] M. Bordemann and J. Hoppe, *The dynamics of relativistic membranes. 1. reduction to two-dimensional fluid dynamics*, *Phys. Lett.* **B 317** (1993) 315 [hep-th/9307036]; *The dynamics of relativistic membranes, 2. Nonlinear waves and covariantly reduced membrane equations*, *Phys. Lett.* **B 325** (1994) 359 [hep-th/9309025].

[6] R. Jackiw and A.P. Polychronakos, *Supersymmetric fluid mechanics*, *Phys. Rev.* **D 62** (2000) 085019.

[7] S. Girvin and T. Jach, *Formalism for the quantum Hall effect: Hilbert space of analytic functions*, *Phys. Rev.* **D 29** (1984) 5617.

[8] G.V. Dunne, R. Jackiw and C.A. Trugenberger, *'Topological' (Chern-Simons) quantum mechanics*, *Phys. Rev.* **D 41** (1990) 661;
G. Dunne and R. Jackiw, *'Peierls substitution' and Chern-Simons quantum mechanics*, *Nucl. Phys.* **33C** *(Proc. Suppl.)* (1993) 114 [`hep-th/9204057`].

[9] Z. Guralnik, R. Jackiw, S-Y. Pi and A.P. Polychronakos, to appear.

[10] A.P. Polychronakos, *Quantum hall states as matrix Chern-Simons theory*, *J. High Energy Phys.* **04** (2001) 011 [`hep-th/0103013`].

[11] F. Calogero, *Solution of the one-dimensional N body problems with quadratic and/or inversely quadratic pair potentials*, *J. Math. Phys.* **12** (1971) 419.

[12] B. Sutherland, *Exact results for a quantum many body problem in one- dimension*, *Phys. Rev.* **A 4** (1971) 2019; *Exact results for a quantum many body problem in one-dimension. 2*, *Phys. Rev.* **A 5** (1972) 1372.

[13] J. Moser, *Three integrable hamiltonian systems connected with isospectral deformations*, *Adv. Math.* **16** (1975) 1.

[14] M.A. Olshanetsky and A.M. Perelomov, *Classical integrable finite dimensional systems related to lie algebras*, *Phys. Rep.* **71** (1981) 313; *Quantum integrable systems related to Lie algebras*, *Phys. Rep.* **94** (1983) 6.

[15] For a review closest in spirit to the present discussion, see A.P. Polychronakos, *Generalized statistics in one dimension*, published in *Topological aspects of low-dimensional systems*, Les Houches session LXIX (1998), Springer, [`hep-th/9902157`].

[16] A.P. Polychronakos, *Nonrelativistic bosonization and fractional statistics*, *Nucl. Phys.* **B 324** (1989) 597.

[17] J.M. Leinaas and J. Myrheim, *Intermediate statistics for vortices in superfluid helium*, *Phys. Rev.* **B 37** (1988) 9286.

[18] A.P. Polychronakos, *Exact anyonic states for a general quadratic hamiltonian*, *Phys. Lett.* **B 264** (1991) 362.

[19] E. Westerberg and T.H. Hansson, *Quantum mechanics on thin cylinders*, `cond-mat/9301018`.

[20] L. Brink, T.H. Hansson, S. Konstein and M.A. Vasiliev, *The Calogero model: anyonic representation, fermionic extension and supersymmetry*, *Nucl. Phys.* **B 401** (1993) 591 [`hep-th/9302023`].

[21] S. Ouvry, *On the relation between the anyon and Calogero models*, `cond-mat/9907239`.

[22] H. Azuma and S. Iso, *Explicit relation of quantum Hall effect and Calogero-Sutherland model*, *Phys. Lett.* **B 331** (1994) 107 [hep-th/9312001].

[23] S. Iso and S.J. Rey, *Collective field theory of the fractional quantum Hall edge state and the Calogero-Sutherland model*, *Phys. Lett.* **B 352** (1995) 111 [hep-th/9406192].

[24] S. Hellerman and M.V. Raamsdonk, *Quantum Hall physics equals noncommutative field theory*, hep-th/0103179.

[25] A.P. Polychronakos, *Noncommutative Chern-Simons terms and the noncommutative vacuum*, *J. High Energy Phys.* **11** (2000) 008 [hep-th/0010264].

[26] V.P. Nair and A.P. Polychronakos, *On level quantization for the noncommutative Chern-Simons theory*, hep-th/0102181.

[27] J.A. Harvey, *Topology of the gauge group in noncommutative gauge theory*, hep-th/0105242.

[28] A.P. Polychronakos, *Flux tube solutions in noncommutative gauge theories*, *Phys. Lett.* **B 495** (2000) 407 [hep-th/0007043].

[29] D.J. Gross and N.A. Nekrasov, *Dynamics of strings in noncommutative gauge theory*, *J. High Energy Phys.* **10** (2000) 021 [hep-th/0007204]; *Solitons in noncommutative gauge theory*, *J. High Energy Phys.* **03** (2001) 044 [hep-th/0010090].

[30] M. Aganagic, R. Gopakumar, S. Minwalla and A. Strominger, *Unstable solitons in noncommutative gauge theory*, *J. High Energy Phys.* **04** (2001) 001 [hep-th/0009142].

[31] J.A. Harvey, P. Kraus and F. Larsen, *Exact noncommutative solitons*, *J. High Energy Phys.* **12** (2000) 024 [hep-th/0010060].

[32] X.G. Wen, *Chiral luttinger liquid and the edge excitations in the fractional quantum Hall states*, *Phys. Rev.* **B 41** (1990) 12838.

[33] S. Iso, D. Karabali and B. Sakita, *One-dimensional fermions as two-dimensional droplets via Chern-Simons theory*, *Nucl. Phys.* **B 388** (1992) 700 [hep-th/9202012]; *Fermions in the lowest landau level: bosonization, W_∞ algebra, droplets, chiral bosons*, *Phys. Lett.* **B 296** (1992) 143 [hep-th/9209003].

[34] A. Cappelli, C.A. Trugenberger and G.R. Zemba, *Infinite symmetry in the quantum Hall effect*, *Nucl. Phys.* **B 396** (1993) 465 [hep-th/9206027].

[35] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, *M-theory as a matrix model: a conjecture*, *Phys. Rev.* **D 55** (1997) 5112 [hep-th/9610043].

[36] I.W. Taylor, *D-brane field theory on compact spaces*, *Phys. Lett.* **B 394** (1997) 283 [hep-th/9611042].

[37] A.P. Polychronakos, *Unitary matrix model for toroidal compactifications of M-theory*, *Phys. Lett.* **B 403** (1997) 239 [hep-th/9703073].

[38] B. Morariu and A. P. Polychronakos, to appear.

[39] J.A. Minahan and A.P. Polychronakos, *Interacting fermion systems from two-dimensional QCD*, *Phys. Lett.* **B 326** (1994) 288 [hep-th/9309044].

[40] A.P. Polychronakos, *Integrable systems from gauged matrix models*, *Phys. Lett.* **B 266** (1991) 29.

[41] A. Gorsky and N. Nekrasov, *Hamiltonian systems of Calogero type and two-dimensional Yang-Mills theory*, *Nucl. Phys.* **B 414** (1994) 213 [hep-th/9304047].

[42] A.P. Polychronakos, *Waves and solitons in the continuum limit of the Calogero-Sutherland model*, *Phys. Rev. Lett.* **74** (1995) 5153 [hep-th/9411054].

[43] D. Bak, K. Lee and J.-H. Park, *Chern-Simons theories on noncommutative plane*, hep-th/0102188.

[44] Y. Nambu in *From* SU(3) *to gravity*, pp 45-52, Gotsman E. ed., Tauber G. ed.; Y. Nambu, B. Bambah and M. Gross, *One-dimensional quark gas*, *Phys. Rev.* **D 26** (1982) 2875.

[45] D. Karabali and B. Sakita, to appear.